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Some applications of spectral theory of nonnegative matrices to input–output models

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Abstract

We show the necessary and sufficient condition that a nonnegative matrix has a unique positive eigenvector, where the analytic expression displaying the linear relations between each remnant component and a *basic characteristic subvector* of the unique eigenvector is discovered when the nonnegative matrix is reducible. As a result, we infer the exact necessary and sufficient condition that the iteration matrix $M^{-1}N$ as a special nonnegative matrix has a unique positive eigenvector when $M - N$ is an M-splitting, which is applied to the condition for the existence and uniqueness of a balanced growth solution for the Leontief dynamic input–output model. Previous work in the field did not clearly involve the uniqueness of the balanced growth solution. In this paper we develop the prior results. That is, we find the necessary and sufficient condition that the Leontief dynamic input–output model has a unique balanced growth solution. Finally, we obtain the necessary and sufficient condition for the existence and uniqueness of both the balanced growth solution and the production prices system. © 2001 Elsevier Science Inc. All rights reserved.

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1. Introduction

Frobenius [3, Section 11] first discovered the existence of the semipositive eigenvector of a nonnegative matrix, though his results were not stated according to the current form. Schneider [7] and Carlson [2] discussed this subject in terms of an *M*-matrix. More recently, Victory [12] generalized Frobenius' results by employing the graph theoretic concepts. As a summary, Schneider [9] surveyed these issues. In this paper, using the above known results, we show the necessary and sufficient condition that a nonnegative matrix T has a unique positive (right) eigenvector R, where the uniqueness and the reducible case are emphasized. We find that if T is reducible, then R is based on a *basic characteristic subvector* R_1 which is a unique positive eigenvector corresponding to the only one basic class of T, and each remnant component of R is the linear function of the component(s) of R_1. The motivation comes from some input–output economic problems in which a unique positive eigenvector of a nonnegative matrix has to be solved.

Schneider [8] introduced the definition of an *M*-splitting of a real matrix $A = M - N$ and investigated the spectral properties of the iteration matrix $M^{-1}N$ by considering the relationships of the graphs of A, M, N, and $M^{-1}N$, where M is a nonsingular *M*-matrix and N is a nonnegative matrix. Starting with Schneider's results, we deduce the elaborate necessary and sufficient condition that the iteration matrix $M^{-1}N$ as a special nonnegative matrix has a unique positive (right) eigenvector, which is motivated by the applications of $M^{-1}N$ to the dynamic input–output model. The dual case can be easy obtained from the above condition.

Szyld [10,11] studied the conditions for the existence of a balanced growth solution for the Leontief dynamic input–output model. Besides, Marek and Szyld [5] generalized both Schneider's results in [8] and Szyld's results in [10,11]. We develop these results to yield the theorem on the uniqueness of the balanced growth solution.

In Section 2, the necessary and sufficient condition that a nonnegative matrix T has a unique positive eigenvector R is revealed, where if T is reducible, then a unique positive eigenvector of the irreducible principal square submatrix corresponding to the only one basic class of T associated with a normal form is called a *basic characteristic subvector* of T since it is the basis of R. The definition of this new concept is mainly due to the discovery of the analytic expression $(2.4)_i$, which displays the linear relations between each remnant component and the basic characteristic subvector of R.

In Section 3, applying the results of Section 2 and [8], we infer the accurate necessary and sufficient condition that the iteration matrix $M^{-1}N$ has a unique positive right eigenvector when $M - N$ is an *M*-splitting. As its dual form, the necessary and sufficient condition that the iteration matrix NM^{-1} has a unique positive left eigenvector when $M - N$ is an *M*-splitting is simply noted without details. So the necessary and sufficient condition that both $M^{-1}N$ has a unique positive right eigenvector and NM^{-1} has a unique positive left eigenvector is obtained. Section 3 is the foundation of Sections 4 and 5.

In Section 4, we first present the definition of the uniqueness of a balanced growth solution for the Leontief dynamic input–output model. Employing the outcome in Section 3, we derive the elaborate necessary and sufficient condition for the existence and uniqueness of a balanced growth solution for the Leontief dynamic input–output model, whose economic meaning is very exact.

Section 5 is an economic consequence of Corollary 3.1.

Let $\wedge$, $\Rightarrow$ and $\Leftrightarrow$ denote conjunction, implication and equivalency, respectively. Let Ø stand for the empty set. Let 0 be zero or zero vector or zero matrix. The vector or matrix $A > 0$ means that A is *semipositive*, i.e., each entry of A is nonnegative, and at least one entry is positive. The vector or matrix $A \gg 0$ means that A is *positive*, i.e., each entry of A is positive. By A^{t} we indicate the transpose of vector or matrix A. Let $\rho(A)$ be the spectral radius of matrix A. The unit matrix is symbolized by I. The meaning that a square matrix A has a *unique* eigenvector corresponding to the eigenvalue λ, or the vector is a *unique* eigenvector of a square matrix A associated with the eigenvalue λ, is that the dimension of the eigenspace A_λ is one.

2. Necessary and sufficient condition that the nonnegative matrix has a unique positive eigenvector

In this section we always assume without loss of generality that a nonnegative $n \times n$ matrix T has a (lower triangular) Frobenius normal form

$$T = \begin{bmatrix} T_{11} & & & 0 \\ T_{21} & T_{22} & & \\ \vdots & \vdots & \ddots & \\ T_{r1} & T_{r2} & \cdots & T_{rr} \end{bmatrix}. \tag{2.1}$$

Lemma 2.1. *Let R be a semipositive (right) eigenvector of T associated with $\rho(T)$. For the following $2r+2$ conditions:*

(i_i) $\forall j \in \{1, \ldots, i-1, i+1, \ldots, r\}$, $\quad \rho(T) = \rho(T_{ii}) > \rho(T_{jj})$ *for* $i = 1, 2, \ldots, r$;

(ii_i) *R has a subvector R_i which is a unique positive eigenvector of T_{ii}, and each remnant positive component (if there exists) of R is the linear function of the component(s) of R_i for* $i = 1, 2, \ldots, r$;

(iii) *T has only one final class*;

(iv) *R is positive and unique*;

we have

(1) (i_i) $\Rightarrow$ (ii_i), $i = 1, 2, \ldots, r$;

(2) $[(\mathrm{i}_1) \wedge (\mathrm{iii})] \Leftrightarrow$ (iv).

Proof. Let $\lambda = \rho(T)$. As is well known, the semipositive eigenvector $R = (R_1^{\mathrm{t}}, R_2^{\mathrm{t}}, \ldots, R_r^{\mathrm{t}})^{\mathrm{t}}$ exists, where R_i is the subvector of R corresponding to formula (2.1) for $i = 1, 2, \ldots, r$.

We prove (1). Obviously, (i_i) implies $r \geqslant 2$. By [4, Lemma 6.2], $R_m = 0$ $(m = 1, \ldots, i-1)$, and R_i is a unique positive eigenvector of T_{ii} associated with $\lambda = \rho(T_{ii})$ since $T_{ii} > 0$ is irreducible. Moreover, if $i < r$, then

$$R_k = (\lambda I_k - T_{kk})^{-1} \sum_{j=i}^{k-1} T_{kj} R_j \qquad (k = i+1, \ldots, r), \tag{2.2}$$

where I_i is the proper identity matrix for $i = 1, 2, \ldots, r$, and by [1, (2.7) Theorem, p.141] $(\lambda I_k - T_{kk})^{-1} \gg 0$ since T_{kk} is irreducible or a nonnegative 1×1 matrix for $k = i+1, \ldots, r$. Next we prove that if $i < r$, then each component of R_k is the linear function of the component(s) of R_i for $k = i+1, \ldots, r$.

If $i < r$, by formula (2.2), we have

$$R_{i+1} = (\lambda I_{i+1} - T_{i+1i+1})^{-1} T_{i+1i} R_i. \qquad (2.3)_i$$

If $i < r-1$, let $b_0 = i$, we still require proving

$$\begin{aligned} R_a = {} & (\lambda I_a - T_{aa})^{-1} \\ & \left\{ T_{ai} + \sum_{j=1}^{a-1-i} \left[\sum_{b_1=i+1}^{a-j} \cdots \sum_{b_j=b_{j-1}+1}^{a-1} T_{ab_j} \right.\right. \\ & \left.\left. \times \prod_{x=j}^{1} (\lambda I_{b_x} - T_{b_x b_x})^{-1} T_{b_x b_{x-1}} \right] \right\} R_i \quad (a = i+2, \ldots, r) \end{aligned} \qquad (2.4)_i$$

by means of mathematical induction on r.

Let $F_k = (\lambda I_k - T_{kk})^{-1}$, $k = i+1, \ldots, r$. By formulas (2.2) and $(2.3)_i$ we have

$$\begin{aligned} R_{i+2} &= F_{i+2}(T_{i+2i} R_i + T_{i+2i+1} R_{i+1}) \\ &= F_{i+2}(T_{i+2i} + T_{i+2i+1} F_{i+1} T_{i+1i}) R_i, \end{aligned}$$

i.e., formula $(2.4)_i$ holds if $r = i+2$. Suppose that formula $(2.4)_i$ holds if $r = m$, i.e.,

$$\begin{aligned} R_a = F_a \left[T_{ai} + \sum_{j=1}^{a-1-i} \left(\sum_{b_1=i+1}^{a-j} \cdots \sum_{b_j=b_{j-1}+1}^{a-1} T_{ab_j} \prod_{x=j}^{1} F_{b_x} T_{b_x b_{x-1}} \right) \right] R_i \\ (a = i+2, \ldots, m). \end{aligned} \tag{2.5}$$

We only need to prove

$$R_{m+1} = F_{m+1} \left[T_{m+1i} + \sum_{j=1}^{m-i} \left(\sum_{b_1=i+1}^{m+1-j} \cdots \sum_{b_j=b_{j-1}+1}^{m} T_{m+1b_j} \prod_{x=j}^{1} F_{b_x} T_{b_x b_{x-1}} \right) \right] R_i.$$

By formulas (2.2) and $(2.3)_i$, we have

$$R_{m+1} = F_{m+1}\left(T_{m+1i}R_i + T_{m+1i+1}F_{i+1}T_{i+1i}R_i + \sum_{a=i+2}^{m} T_{m+1a}R_a\right).$$

Therefore, by formula (2.5), we only require proving

$$\begin{aligned}&\sum_{a=i+2}^{m} T_{m+1a}F_a \sum_{j=1}^{a-1-i}\left(\sum_{b_1=i+1}^{a-j}\cdots\sum_{b_j=b_{j-1}+1}^{a-1} T_{ab_j}\prod_{x=j}^{1} F_{b_x}T_{b_xb_{x-1}}\right)\\ &\quad=\sum_{j=2}^{m-i}\left(\sum_{b_1=i+1}^{m+1-j}\cdots\sum_{b_j=b_{j-1}+1}^{m} T_{m+1b_j}\prod_{x=j}^{1} F_{b_x}T_{b_xb_{x-1}}\right).\end{aligned} \tag{2.6}$$

Let

$$Y(h_e, h_{e-1}, \ldots, h_1) = T_{m+1h_e}F_{h_e}T_{h_eh_{e-1}}\cdots F_{h_2}T_{h_2h_1}F_{h_1}T_{h_1i},$$

where $i + 1 \leqslant h_1$, $h_{s-1} + 1 \leqslant h_s$, $s = 2, \ldots, e$, $h_e \leqslant m$, $2 \leqslant e \leqslant m - i$. Then formula (2.6) is equivalent to

$$\begin{aligned}&\sum_{a=i+2}^{m}\sum_{j=1}^{a-1-i}\sum_{b_1=i+1}^{a-j}\cdots\sum_{b_j=b_{j-1}+1}^{a-1} Y(a, b_j, \ldots, b_1)\\ &\quad=\sum_{j=2}^{m-i}\sum_{b_1=i+1}^{m+1-j}\cdots\sum_{b_j=b_{j-1}+1}^{m} Y(b_j, b_{j-1}, \ldots, b_1).\end{aligned} \tag{2.7}$$

It is not difficult to prove formula (2.7). Hence formula $(2.4)_i$ holds. Formulas $(2.3)_i$ and $(2.4)_i$ show that if $i < r$, then each component of R_k is the linear function of the component(s) of R_i for $k = i + 1, \ldots, r$. The proof of (1) is completed.

We prove (2). Suppose that $(\mathrm{i}_1)\wedge$ (iii) holds. By the proof of (1), (i_1) implies that $R_1 \gg 0$, and λ is a simple root of T. Hence R is a unique eigenvector of T associated with λ. Thus we only require proving $R_k \gg 0$, $k = 2, \ldots, r$. Since $(\lambda I_k - T_{kk})^{-1} \gg 0$, $k = 2, \ldots, r$, the result follows from (iii) and formula (2.2).

Inversely, let (iv) hold. We only need to complete the proof that the reduced graph of $\lambda I - T$ has precisely one singular vertex, which is also the only final vertex. By [9, (3.5) Corollary], since there exists a positive vector R satisfying $(\lambda I - T)R = 0$, the set of singular vertices is equal to the set of final vertices. So each singular vertex is distinguished and hence by [9, (3.1) Theorem], the nullspace of $\lambda I - T$ has a semi-positive basis satisfying [9, (3.2)]. The uniqueness of R means that the dimension of the nullspace of $\lambda I - T$ is one. Thus the above vertex set has only one element. The proof of (2) is completed. □

Remark 2.1. In the proof of result (1), formulas $(2.3)_i$ and $(2.4)_i$ are important. As compared with formula (2.2), formulas $(2.3)_i$ and $(2.4)_i$ thoroughly show the linear relations between each component of R_k and the component(s) of R_i for

$k = i+1, i+2, \ldots, r, i = 1, 2, \ldots, r-1$. Hence R_i plays a basic role in the semipositive eigenvector R.

Definition 2.1. Suppose $r \geqslant 2$ in the normal form (2.1) of T. If $\exists i \in \{1, 2, \ldots, r\}$ such that $\rho(T) = \rho(T_{ii}) > \rho(T_{jj}),\ j = 1, \ldots, i-1, i+1, \ldots, r,$ then a unique positive eigenvector R_i of T_{ii} is called a *basic characteristic subvector* of T corresponding to this normal form.

Theorem 2.1. *Let T be a nonnegative matrix of order n. Then the following conditions are equivalent*:
(i) *T has precisely one basic class*, *which is also the only final class*;
(ii) *T has a unique positive eigenvector R*;
where if T is reducible, *then T corresponding to the normal form* (2.1) *has a basic characteristic subvector* R_1, *and each remnant component of R is both positive and the linear function of the component*(*s*) *of* R_1, *whose analytic expression is formula* $(2.3)_1$ *or* $(2.4)_1$.

3. Spectral properties of the iteration matrices

In this section we always assume that the $n \times n$ real matrix $M - N$ is a nontrivial M-splitting, i.e., M is a nonsingular M-matrix and N is a semipositive square matrix. Also, we assume without loss of generality that $M - N$ has a Frobenius normal form

$$M - N = \begin{bmatrix} M_{11} - N_{11} & & & 0 \\ M_{21} - N_{21} & M_{22} - N_{22} & & \\ \vdots & \vdots & \ddots & \\ M_{r1} - N_{r1} & M_{r2} - N_{r2} & \cdots & M_{rr} - N_{rr} \end{bmatrix}. \tag{3.1}$$

Thus, similarly to formula (3.1), $M^{-1}N$ has the corresponding (lower triangular) partition, where the main diagonal block $M_{kk}^{-1}N_{kk}$ may be reducible by [8, Lemma 3.4] for $k = 1, \ldots, r$.

Lemma 3.1. *If* $M - N$ *that has only one final class is reducible*, *and* $\rho(M_{11}^{-1}N_{11}) > \rho(M_{ii}^{-1}N_{ii}), i = 2, \ldots, r,$ *then* $M^{-1}N$ *is reducible and it has exactly one basic class*, *which is also the only final class.*

Proof. From [6, Theorem 4.3, p.160], M_{11} is an M-matrix. Thus $M_{11} - N_{11}$ is a nontrivial M-splitting, and $M_{11}^{-1}N_{11}$ and $M^{-1}N$ have the same basic class(es) since $\rho(M_{11}^{-1}N_{11}) > \rho(M_{ii}^{-1}N_{ii}), i = 2, \ldots, r$. By [8, Lemma 3.4], $M_{11}^{-1}N_{11}$ has exactly one basic class, which is also the only final class of $M_{11}^{-1}N_{11}$. Hence $M^{-1}N$ has exactly one basic class, which is also a final class of $M^{-1}N$. To establish that it is the only final class of $M^{-1}N$, let p be any element in the final class of $M_{11}^{-1}N_{11}$. Since this class is basic, p is a vertex of a nonempty circuit of the graph $\Gamma(M^{-1}N)$. For $\forall q \in \{1, 2, \ldots, n\}$, q must have access to p in the graph $\Gamma(M - N)$, since $M - N$ has only one final class. The result now follows from [8, Theorem 2.8]. $\square$

Lemma 3.2. *If $M - N$ that has only one final class is reducible, and $\exists j \in \{2, \ldots, r\}$ such that $\rho(M_{11}^{-1}N_{11}) \leqslant \rho(M_{jj}^{-1}N_{jj})$, then*
(i) *$M^{-1}N$ has at least two basic classes;*
or
(ii) *$M^{-1}N$ has at least two final classes;*
or
(iii) *a basic class of $M^{-1}N$ is not final.*

Proof. Suppose that conclusions (i) and (ii) do not hold. Then $M_{11}^{-1}N_{11}$ must contain the only final class of $M^{-1}N$. Since $\rho(M_{11}^{-1}N_{11}) \leqslant \rho(M_{jj}^{-1}N_{jj})$, $M_{11}^{-1}N_{11}$ does not contain the only basic class of $M^{-1}N$. Thus conclusion (iii) holds. $\square$

From [8, Lemma 2.4 and Corollary 2.6], we can easy observe that if $M - N$ has at least two final classes then $M^{-1}N$ has also at least two final classes.

Theorem 3.1. *$M^{-1}N$ has a unique positive (right) eigenvector R if and only if*
(1) *$M - N$ is irreducible, where*
  (i) *if each column of N has at least one positive entry, then $M^{-1}N$ is irreducible;*
  (ii) *if N has at least one entire column of zeros, then the reducible $M^{-1}N$ has a basic characteristic subvector R_1 corresponding to the only basic class of $M^{-1}N$, and each remnant component of R is the linear function of the component(s) of R_1;*
or
(2) *$M - N$ that has only one final class is reducible, and $\rho(M_{11}^{-1}N_{11}) > \rho(M_{ii}^{-1}N_{ii})$, $i = 2, \ldots, r$, where the reducible $M^{-1}N$ has a basic characteristic subvector $\overline{R}_1$ corresponding to the only basic class of both $M_{11}^{-1}N_{11}$ and $M^{-1}N$, and each remnant component of R is the linear function of the component(s) of $\overline{R}_1$.*

Proof. From [8, Lemma 3.4], Lemma 3.1 and Theorem 2.1, we can obtain “If”. From Lemma 3.2 and the above observation, as well as Theorem 2.1, we can obtain “Only If”. $\square$

We do not state the dual form of Theorem 3.1 since it is easy to obtain the dual case from this theorem, where the iteration matrix becomes NM^{-1}, whose left eigenvector is considered.

Corollary 3.1. *Both $M^{-1}N$ has a unique positive right eigenvector R and NM^{-1} has a unique positive left eigenvector L if and only if $M - N$ is irreducible, where*

(i) *if each column of N has at least one positive entry, then $M^{-1}N$ is irreducible;*

(ii) *if each row of N has at least one positive entry, then NM^{-1} is irreducible;*

(iii) *if N has at least one entire column of zeros, then the reducible $M^{-1}N$ has a basic right characteristic subvector R_1 corresponding to the only basic class of $M^{-1}N$, and each remnant component of R is the linear function of the component(s) of R_1;*

(iv) *if N has at least one entire row of zeros, then the reducible NM^{-1} has a basic left characteristic subvector L_2 corresponding to the only basic class of NM^{-1}, and each remnant component of L is the linear function of the component(s) of L_2.*

Proof. Since NM^{-1} is the dual form of $M^{-1}N$, "If" is clear. We prove "Only If". If $M - N$ is reducible, $\rho(M_{11}^{-1}N_{11}) > \rho(M_{rr}^{-1}N_{rr})$ and $\rho(M_{rr}^{-1}N_{rr}) = \rho(N_{rr}M_{rr}^{-1}) > \rho(N_{11}M_{11}^{-1}) = \rho(M_{11}^{-1}N_{11})$ are contradictory. □

4. Necessary and sufficient condition that the Leontief dynamic input–output model has a unique balanced growth solution

It is known that if no change in the technology is assumed over time, then both discrete and closed Leontief dynamic input–output model of an economy is

$$(I - A)X_k = B(X_{k+1} - X_k), \tag{4.1}$$

where $A > 0$ and $B > 0$ are the intermediate input coefficient matrix and the capital input coefficient matrix, respectively, $\rho(A) < 1$ hence $(I - A)^{-1} > 0$ exists, and X_k is the column vector of gross output at time period *k*. An important solution of this model is the so-called "balanced growth solution (BGS)", i.e., this solution means that the gross output of each sector increases by a constant percentage per unit of time, the mutual proportions in which various sectoral products are produced remain constant, i.e.,

$$X_k = (1 + \delta)^k X \gg 0, \tag{4.2}$$

where $\delta > 0$ is called the *balanced growth rate* of the economy system, and X is a column vector of gross output.

Definition 4.1. Let the set H = {positive vector $X_k = (1 + \delta)^k X | (I - A)X_k = B(X_{k+1} - X_k)$}.

(i) If $H \neq \emptyset$, then model (4.1) is called to have a *BGS* $X_k = (1 + \delta)^k X$, or $X_k = (1 + \delta)^k X$ is said to be a *BGS* of model (4.1).

(ii) If $\exists X_k^* = (1+\delta)^k X^* \in H$, such that $(\forall X_k \in H \Rightarrow X_k = \alpha X_k^*)$ holds, where α is a positive number, then model (4.1) is called to have a *unique BGS* $X_k^* = (1+\delta)^k X^*$, or $X_k^* = (1+\delta)^k X^*$ is said to be a *unique BGS* of model (4.1).

Example 4.1. Let the intermediate input coefficient matrix and the capital input coefficient matrix of an economy be respectively

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0.3 & 0 & 0.2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2.5 & 0 \\ 0.4 & 0 & 0.8 \end{bmatrix}.$$

Then $X_k = 1.2^k(32, \sigma, 19)^{\mathrm{t}}$ satisfies model (4.1). Thus, model (4.1) of this economy has a BGS $X_k = 1.2^k(32, \sigma, 19)^{\mathrm{t}}$, and the balanced growth rate of the economy is 0.2. Since σ can be an arbitrary positive number, $X_k = 1.2^k(32, \sigma, 19)^{\mathrm{t}}$ is not a unique BGS, i.e., model (4.1) of this economy has infinitely many BGSs.

Example 4.2. Let the intermediate input coefficient matrix and the capital input coefficient matrix of an economy be respectively

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0.1 & 0 & 0 \\ 0 & 0 & 0.2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 0 & 0 \\ 0.4 & 0 & 0.8 \end{bmatrix}.$$

Then $X_k = 1.2^k(40, 4, 5)^{\mathrm{t}}$ satisfies model (4.1). Thus model (4.1) of this economy has a BGS $X_k = 1.2^k(40, 4, 5)^{\mathrm{t}}$, and the balanced growth rate of the economy is 0.2. Afterward we shall prove that $X_k = 1.2^k(40, 4, 5)^{\mathrm{t}}$ is a unique BGS.

Clearly, formula (4.2) satisfies model (4.1) if and only if $(I-A)^{-1}BX = (1/\delta)X$, where $\delta = 1/\rho[(I-A)^{-1}B]$ by [1, (1.12) Corollary, p.28]. Thus we have:

Proposition 4.1. *Model* (4.1) *of an economy has a* (*unique*) *BGS if and only if the semipositive square matrix* $(I-A)^{-1}B$ *has a* (*unique*) *positive eigenvector.*

Proof. We only require proving the case of uniqueness.

$$\begin{aligned}
&[\text{Model (4.1) has a unique BGS}] \\
&\quad \Leftrightarrow [\exists X_k^* = (1+\delta)^k X^* \in H \text{ such that } (\forall X_k \in H \Rightarrow X_k = \alpha X_k^*)] \\
&\quad \Leftrightarrow \{X_k^* = (1+\delta)^k X^* \text{satisfies model (4.1)}, \\
&\qquad\quad \text{and } [(\text{formula (4.2) satisfies model (4.1)}) \\
&\qquad \Rightarrow (1+\delta)^k X = \alpha(1+\delta)^k X^*]\} \\
&\quad \Leftrightarrow \{(I-A)^{-1}BX^* = (1/\delta)X^*, \\
&\qquad\quad \text{and } [(I-A)^{-1}BX = (1/\delta)X \Rightarrow X = \alpha X^*]\} \\
&\quad \Leftrightarrow [(I-A)^{-1}\, B \text{ has a unique positive eigenvector}]. \qquad \square
\end{aligned}$$

From Proposition 4.1, when $(I-A)^{-1}B$ has a positive eigenvector X we can adjust X_0, the column vector of the initial gross output, such that $X_0=\alpha X$, where α is a positive number, then X_0 becomes a configuration vector of gross output that enables the economy to grow at the balanced growth rate $1/\rho[(I-A)^{-1}B]$. For convenience sake we call the readjusted X_0 to be a *balanced growth configuration vector* of the economy.

We consider Examples 4.1 and 4.2 again. For Example 4.1,

$$(I-A)^{-1}B=\begin{bmatrix}5 & 0 & 0\\ 0 & 5 & 0\\ 2.375 & 0 & 1\end{bmatrix}, \quad \rho[(I-A)^{-1}B]=5.$$

Since $(32,\sigma,19)^{\mathrm{t}}$ is a nonunique positive eigenvector of $(I-A)^{-1}B$ associated with 5, $X_k=1.2^k(32,\sigma,19)^{\mathrm{t}}$ is a nonunique BGS of model (4.1), where $X_0=(32,\sigma,19)^{\mathrm{t}}$ is a balanced growth configuration vector of the economy.

For Example 4.2,

$$(I-A)^{-1}B=\begin{bmatrix}5 & 0 & 0\\ 0.5 & 0 & 0\\ 0.5 & 0 & 1\end{bmatrix}, \quad \rho[(I-A)^{-1}B]=5.$$

Since $(I-A)^{-1}B$ has exactly a basic class, which is also the only final class, $(I-A)^{-1}B$ has a unique positive eigenvector $(40,4,5)^{\mathrm{t}}$ by Theorem 2.1. Thus model (4.1) has a unique BGS $X_k=1.2^k(40,4,5)^{\mathrm{t}}$ by Proposition 4.1, where $X_0=(40,4,5)^{\mathrm{t}}$ is a balanced growth configuration vector of the economy.

Szyld researched the conditions for the existence of the BGS, but the uniqueness was not explicitly involved (cf. [10,11]). He first presented the following assumption: "Each column of the matrix B has at least one nonzero entry". Under the hypothesis he proved that "$U=(I-A)^{-1}B$ is irreducible if and only if the sum $C=A+B$ is irreducible". Thus, the BGS exists when $A+B$ is irreducible. In fact, since $(I-A)^{-1}B$ is irreducible, the BGS not only exists but is unique by Theorem 2.1 and Proposition 4.1. Obviously, the condition that $A+B$ is irreducible and each column of the matrix B has at least one nonzero entry is not a necessary, but a sufficient condition that model (4.1) has a unique BGS. For instance,

$$A+B=\begin{bmatrix}5 & 0 & 0\\ 0.1 & 0 & 0\\ 0.4 & 0 & 1\end{bmatrix}$$

is reducible, and each entry of the second column in the matrix B is zero in Example 4.2, but model (4.1) of this economy has a unique BGS.

Moreover, for the reducible $(I-A)^{-1}B$, Szyld [10, Theorem 2] gave a necessary and sufficient condition for the existence of the BGS. Actually, employing the graph theoretic concepts, this theorem is equivalent to [1, (3.10) Theorem, p.40] or [9, (3.5) Corollary]. Evidently, when $g\geqslant 2$ in [10, Theorem 2], i.e., $(I-A)^{-1}B$ has at least two both basic and final classes, $(I-A)^{-1}B$ cannot have a *unique* positive eigenvector by Theorem 2.1. Hence, this necessary and sufficient condition cannot ensure the BGS to be unique. For instance, we consider Example 4.1 again, $(I-A)^{-1}B$ has exactly two both basic and final classes. It is known that model (4.1) of this economy has infinitely many BGSs.

Next, using the new results founded in Section 3 of this paper, we can completely solve the above problems. Namely, we can find the necessary and sufficient condition and its exact economic meaning that the Leontief dynamic input–output model has a unique BGS. Certainly, some stricter restricted conditions, such as "Each column of the matrix B has at least one nonzero entry" and "$A+B$ is irreducible", will be relaxed.

First, we give an economic explanation of the semipositive matrix $U=(I-A)^{-1}B=(u_{ij})_{n\times n}$, where A is the intermediate input coefficient matrix in value terms, and $B=(b_{ij})_{n\times n}$ is the capital input coefficient matrix in value terms. Let $V=(v_1,v_2,\ldots,v_n)$ be the value-added rate (i.e., value-added per unit of gross output value) row vector, and let $G=EB=(g_1,g_2,\ldots,g_n)$ be the capital input rate row vector, where $E=(1,1,\ldots,1)$. Hence

$$g_j=\sum_{i=1}^{n} b_{ij}$$

is the gross capital input per unit of gross output value of sector j for $j=1,2,\ldots,n$. Then $VU=E(I-A)(I-A)^{-1}B=EB=G$, i.e.,

$$\sum_{i=1}^{n} v_i u_{ij}=g_j \quad (j=1,2,\ldots,n).$$

Thus

$$0\leqslant u_{ij}=\frac{\partial g_j}{\partial v_i} \quad (i,j=1,2,\ldots,n),$$

i.e., u_{ij} measures the rate of change of the capital input rate of sector j with respect to a change in the value-added rate of sector i. So $U=(I-A)^{-1}B$ can be called the *linked matrix or multiplier matrix between capital input rate and value-added rate* as with Leontief inverse $(I-A)^{-1}$ can be called the linked matrix or multiplier matrix between gross output and final output.

Obviously, we can directly obtain the necessary and sufficient condition that $(I-A)^{-1}B$ has a unique positive eigenvector from Theorem 2.1, i.e., $(I-A)^{-1}B$ has exactly a basic class, which is also the only final class. The economic meaning of this necessary and sufficient condition, however, is not clear. Hence, in order to find the necessary and sufficient condition that has a both evident and accurate economic interpretation we have to employ Theorem 3.1.

Let $M=I-A$ and $N=B$. Then $M-N$ is irreducible if and only if $A+B$ is irreducible. In economic terms the irreducibility of $A+B$ means that each sector of the economy depends on all others directly or indirectly for either its intermediate products or its capital. Corresponding to a (lower triangular) Frobenius normal form, the reducibility of $A+B$ means that the economy can be divided into $r\geqslant 2$

subeconomies $S_1, S_2, \dots, S_r$ by the interdependence among the intermediate products and the capital, each sector of S_k depends on all others in S_k directly or indirectly for either its intermediate products or its capital, or S_k has only one sector for $k = 1, 2, \dots, r$. Also, if the condition that $A + B$ has only one final class is added, this means that S_i demands neither any intermediate product nor any capital from $S_1, \dots, S_{i-1}$, but S_i supplies either the intermediate products or the capital to at least one subeconomy within $S_1, \dots, S_{i-1}$ for $i = 2, \dots, r$. The economic meaning that each column (row) of B has at least one positive entry is that each sector of the economy demands (supplies) some capital. The economic meaning that B has at least one entire column (row) of zeros is that there exists at least one sector that does not demand (supply) any capital in the economy. Thus, by Proposition 4.1 and Theorem 3.1, we have:

Theorem 4.1. *The Leontief dynamic input–output model* (4.1) *has a unique BGS if and only if*

(1) *each sector of the economy depends on all others directly or indirectly for either its intermediate products or its capital, where*
   (i) *if each sector demands some capital, then the linked matrix between capital input rate and value-added rate,* $(I - A)^{-1}B$, *is irreducible*;
   (ii) *if there exists at least one sector that does not demand any capital, then the reducible* $(I - A)^{-1}B$ *has a basic characteristic subvector which is a subvector of a unique balanced growth configuration vector of the economy, and each remnant component is the linear function of the component*(*s*) *of the subvector*;

or

(2) *the economy can be divided into* $r \geqslant 2$ *subeconomies* $S_1, S_2, \dots, S_r$ *by the interdependence among the intermediate products and the capital, each sector of* S_k *depends on all others in* S_k *directly or indirectly for either its intermediate products or its capital, or* S_k *has only one sector for* $k = 1, 2, \dots, r$, S_i *demands neither any intermediate product nor any capital from* $S_1, \dots, S_{i-1}$, *but* S_i *supplies either the intermediate products or the capital to at least one subeconomy within* $S_1, \dots, S_{i-1}$ *for* $i = 2, \dots, r$, *and* S_1 *as a subeconomy has a unique BGS, whose balanced growth rate is less than that of* S_i *if* S_i *as a subeconomy has also a unique BGS, i.e.,* $\{1/\rho[(I_1 - A_{11})^{-1}B_{11}]\} < \{1/\rho[(I_i - A_{ii})^{-1}B_{ii}]\}$ *if* $\rho[(I_i - A_{ii})^{-1}B_{ii}] > 0$ *for* $i = 2, \dots, r$, *where the reducible* $(I - A)^{-1}B$ *has a basic characteristic subvector corresponding to the only basic class of both* $(I_1 - A_{11})^{-1}B_{11}$ *and* $(I - A)^{-1}B$, *which is a subvector of a unique balanced growth configuration vector of the economy, and each remnant component is the linear function of the component*(*s*) *of the subvector.*

Proof. We only need to prove that S_k as a subeconomy has a unique BGS if and only if $\rho[(I_k - A_{kk})^{-1}B_{kk}] > 0$, $k = 1, 2, \dots, r$. Clearly, the result follows from Proposition 4.1, Theorem 2.1 and [8, Lemma 3.4]. $\square$

5. Necessary and sufficient condition for existence and uniqueness of both BGS and production prices

It is known that the input–output price model of an economy is $P = PA + D$, where $P \gg 0$ is the price row vector, $A > 0$ the physical intermediate input coefficient matrix, and $D > 0$ the row vector of value-added per unit of physical gross output. The so-called "production prices" means that the economy has a uniform capital return rate to each sector, i.e., $P = PA + \varepsilon PB$, which is equivalent to $PB(I - A)^{-1} = (1/\varepsilon)P$, where $B > 0$ is the physical capital input coefficient matrix, $\varepsilon = 1/\rho[B(I - A)^{-1}] > 0$ is called the *uniform capital return rate* of the economy.

As the dual form of Section 4, we have the similar results for the existence and uniqueness of the production prices, where P, $B(I - A)^{-1}$ and ε correspond to X, $(I - A)^{-1}B$ and δ, respectively. Here we do not express these results, but give an economic explanation of the semipositive matrix $W = B(I - A)^{-1} = (w_{ij})_{n\times n}$ that is the dual matrix of $U = (I - A)^{-1}B$. Let $F = (f_1, f_2, \dots, f_n)^{\mathrm{t}}$ be the final demand column vector, $X = (x_1, x_2, \dots, x_n)^{\mathrm{t}}$ the gross output column vector, and $K = BX = (k_1, k_2, \dots, k_n)^{\mathrm{t}}$ the capital supply column vector, where

$$k_i = \sum_{j=1}^{n} b_{ij} x_j$$

is the gross capital supply of sector i, for $i = 1, 2, \dots, n$. Then $WF = B(I - A)^{-1} F = BX = K$, i.e.,

$$\sum_{j=1}^{n} w_{ij} f_j = k_i \quad (i = 1, 2, \dots, n).$$

Hence

$$0 \leqslant w_{ij} = \frac{\partial k_i}{\partial f_j} \quad (i, j = 1, 2, \dots, n),$$

i.e., w_{ij} measures the rate of change of the capital supply of sector i with respect to a change in the final demand of sector j. So $W = B(I - A)^{-1}$ can be called the *linked matrix or multiplier matrix between capital supply and final demand*.

By Corollary 3.1 and Proposition 4.1, we have:

Theorem 5.1. *In an economy both the Leontief dynamic input–output model has a unique BGS and there exists a unique production prices system if and only if each sector of the economy depends on all others directly or indirectly for either its intermediate products or its capital*, *where*

(i) *if each sector demands some capital*, *then the linked matrix between capital input rate and value-added rate*, $(I - A)^{-1}B$, *is irreducible*;

(ii) *if each sector supplies some capital*, *then the linked matrix between capital supply and final demand*, $B(I - A)^{-1}$, *is irreducible*;

(iii) *if there exists at least one sector that does not demand any capital, then the reducible* $(I - A)^{-1}B$ *has a basic right characteristic subvector which is a subvector of a unique balanced growth configuration vector of the economy, and each remnant component is the linear function of the component(s) of the subvector;*

(iv) *if there exists at least one sector that does not supply any capital, then the reducible* $B(I - A)^{-1}$ *has a basic left characteristic subvector which is a subvector of a unique production prices configuration vector of the economy, and each remnant component is the linear function of the component(s) of the subvector.*

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